Cornell Bowers C·IS
Computer Science

Theory Seminar
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Recursive Lattice Reduction

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and

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Lattices

A *lattice* $\mathcal{L} = \mathcal{L}(B)$ is specified by a *basis*
$B = (b_1, b_2, \ldots, b_n)$ of linearly independent vectors $b_i \in \mathbb{R}^d$:

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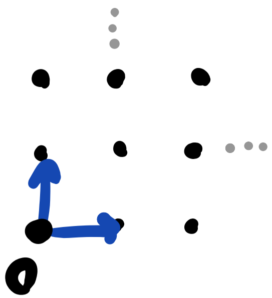
e.g., $\mathbb{Z}^2$: 

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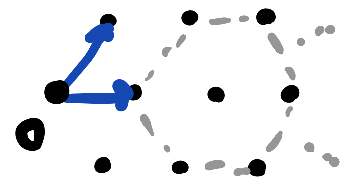
, Hexagonal: 

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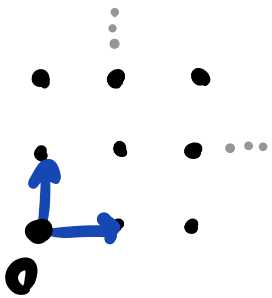
e.g., $\mathbb{Z}^2$: 

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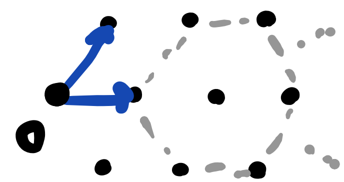
, Hexagonal: 

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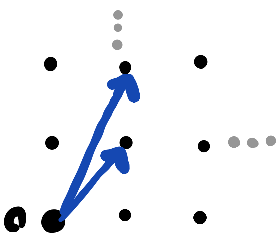
$\mathbb{Z}^2$: 

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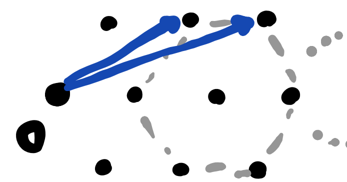
, Hexagonal: 

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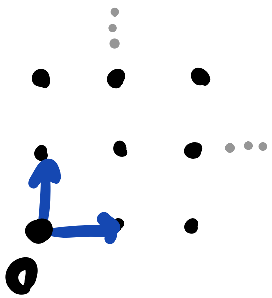
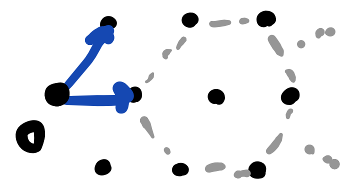
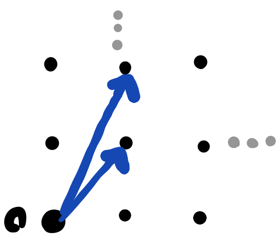
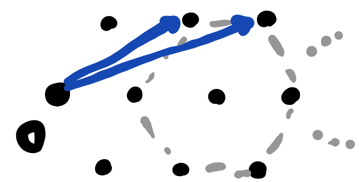
e.g., $\mathbb{Z}^2$:  , $\mathbb{Z}^n$:  , Hexagonal:  , ...

$\mathbb{Z}^2$:  , $\mathbb{Z}^n$:  , Hexagonal:  , ...

n is called the *rank* or *dimension* of $\mathcal{L}$.

Lattices

$$\boxed{\lambda_1(\mathcal{L}) := \min_{\substack{y \in \mathcal{L} \\ y \neq \vec{0}}} \|y\|}$$

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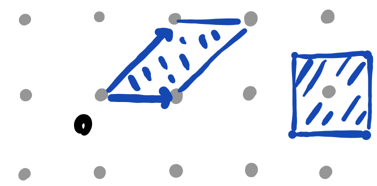

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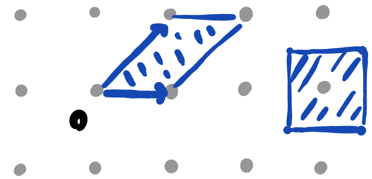

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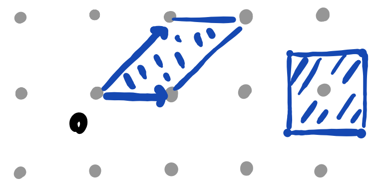

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SVP: Find $v \in \mathcal{L}$ with $\|v\| = \lambda_1(\mathcal{L})$.

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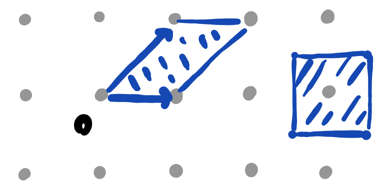

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γ-approximate (H)SVP: Find $v \in \mathcal{L}$:

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Time / Approximation Tradeoffs

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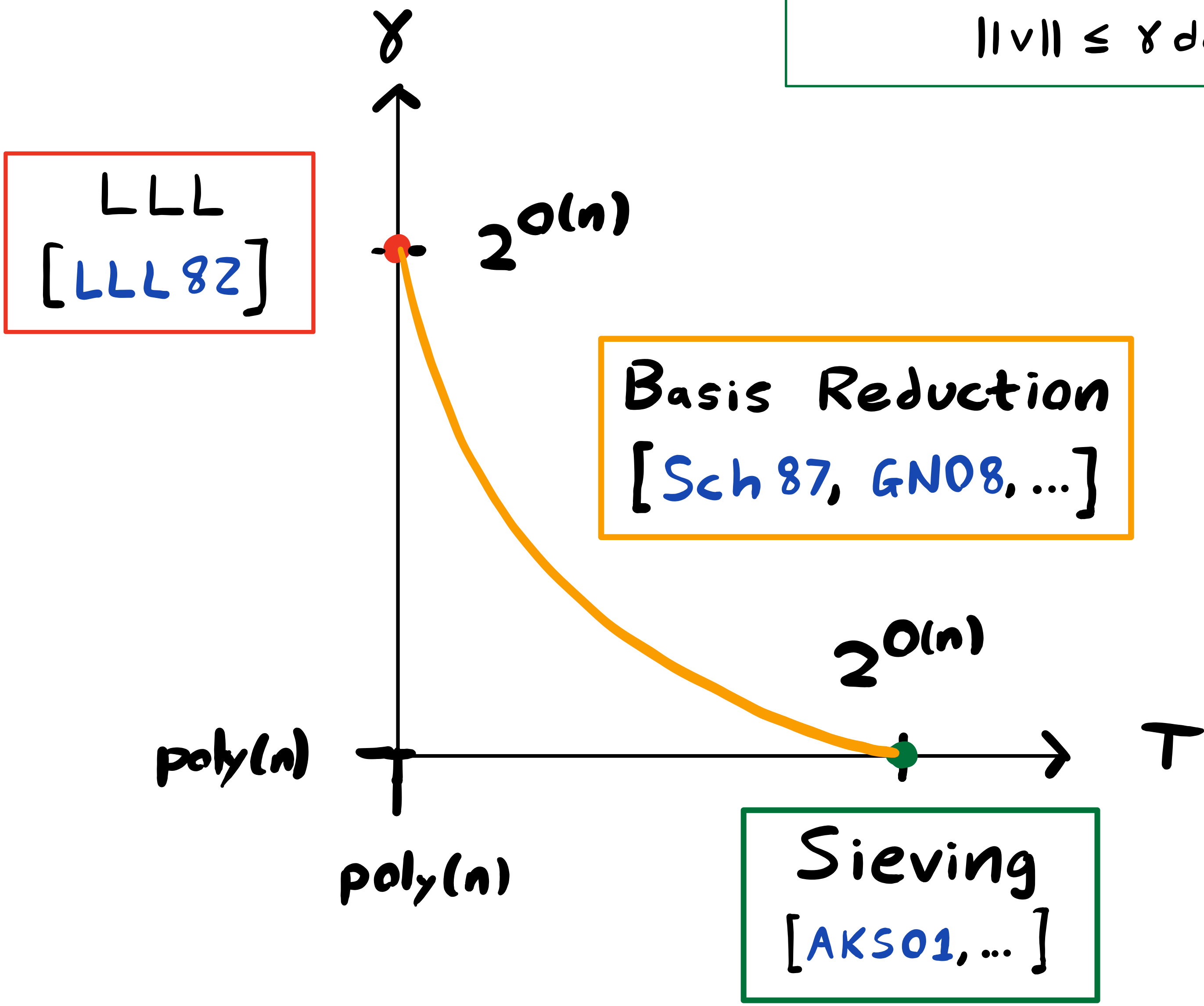

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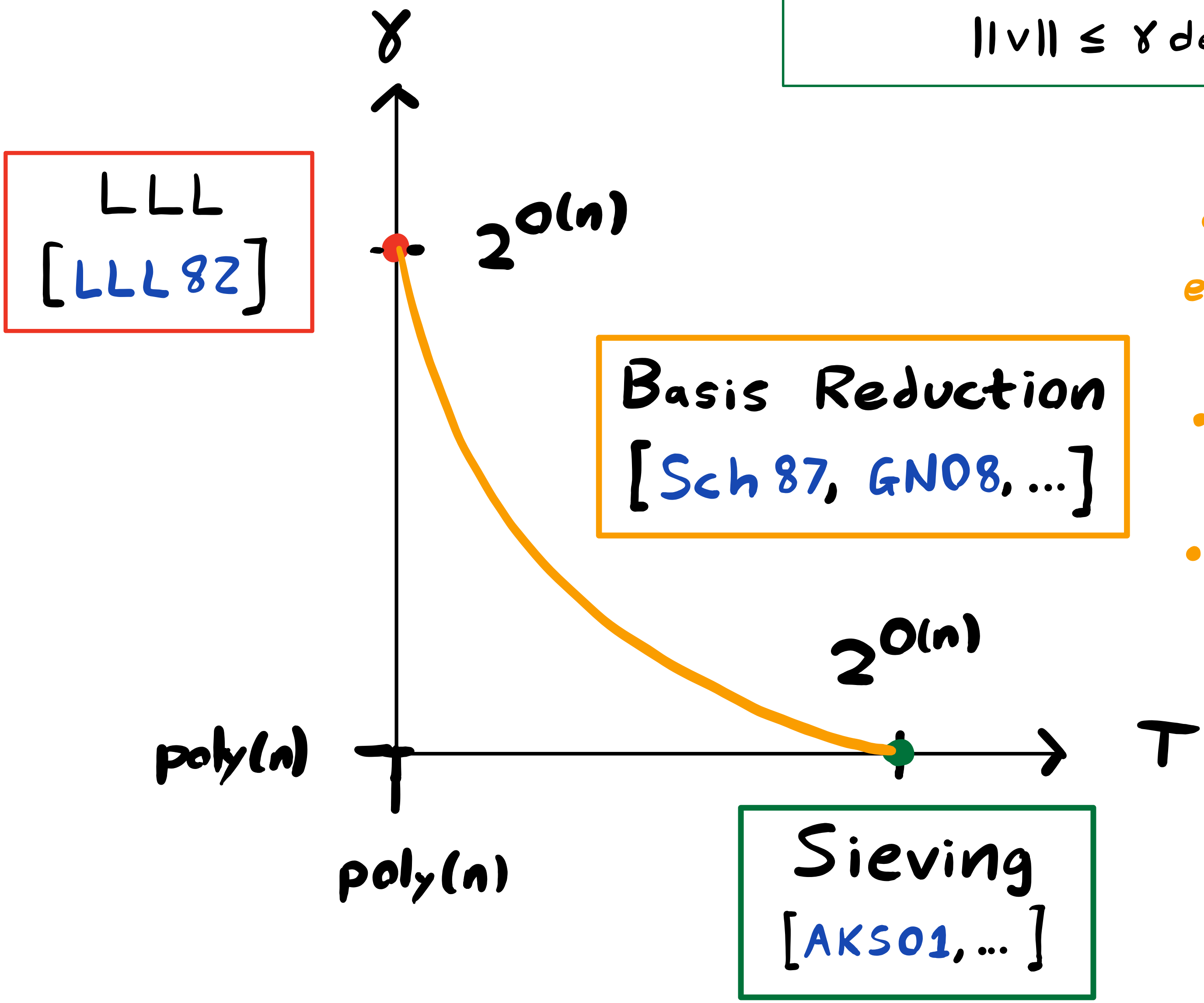

- Works by solving exact SVP in dimension k
- $T = \mathrm{poly}(n)\, 2^{O(k)}$
- $\gamma = k^{\frac{(n-1)}{2(k-1)}} \approx k^{\frac{n}{2k}}$.

Basis Reduction Algorithms

- All basis reduction algorithms follow the basic approach of LLL: iteratively improve the basis by solving SVP exactly in smaller dimension.

- Analysis is intricate
    - Our best (practical) algorithms are *heuristic*

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  - "Recurse on a smaller-dimensional lattice!"

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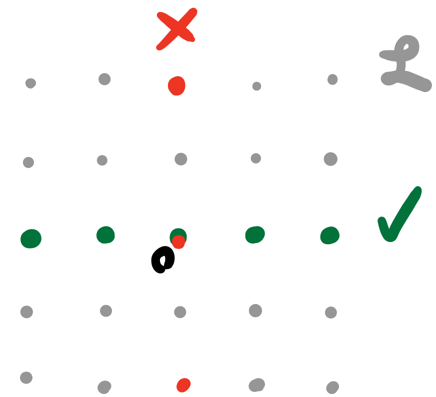
- This talk: match Basis Reduction tradeoff with an "Algorithms 101"-style recursive algorithm.
  - "Recurse on a smaller-dimensional lattice!"
  - Should be a *sublattice*
  - But should still have *short vectors* — i.e., small determinant.

The Densest Sublattice Problem

A sublattice $\mathcal{L}'$ of $\mathcal{L}$ is the intersection of $\mathcal{L}$ with a subspace.



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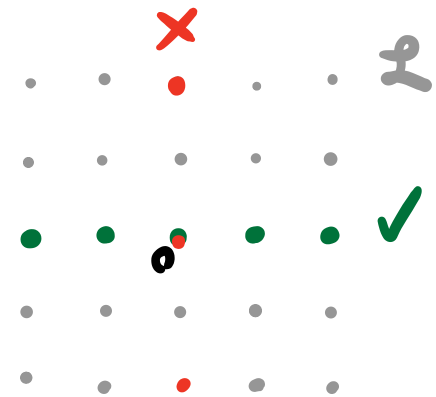

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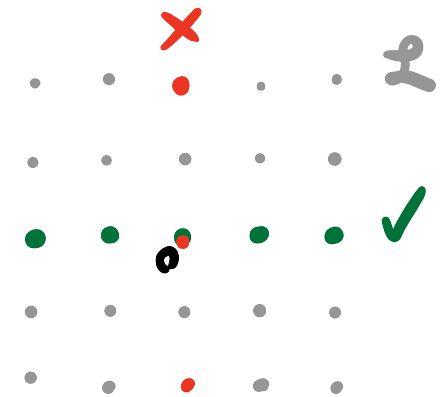

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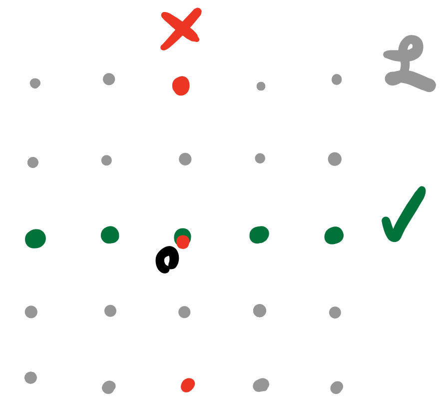

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γ-DSP with $\ell = 1$ is exactly γ-SVP.

An approach for solving SVP

$A(\mathcal{L})$:

1. Find dense sublattice $\mathcal{L}' \subset \mathcal{L}$
   (somehow)

2. Return $A(\mathcal{L}')$.

- For the base case, when $\mathrm{rank}(\mathcal{L}) = k$,
  output $SVP(\mathcal{L})$ — that is, use an exact algorithm.

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<u>Proof.</u> $\det(\mathcal{L}'') \leq \gamma_1 \cdot \det(\mathcal{L}')^{\ell/m}$

$$= \gamma_1 \cdot \left(\gamma_2 \det(\mathcal{L})^{\ell/m}\right)^{m/n} = \gamma_1 \cdot \gamma_2^{\ell/m} \det(\mathcal{L})^{\ell/m}.$$

γ-DSP is Self-Dual

$$\boxed{\mathcal{L}^* := \{ w \in \mathrm{span}(\mathcal{L}) : \forall y \in \mathcal{L}, \langle w, y \rangle \in \mathbb{Z} \}}$$

$$\boxed{(\mathcal{L}^*)^* = \mathcal{L}} \qquad \boxed{\det(\mathcal{L}^*) = 1/\det(\mathcal{L})}$$

- There is a bijection from rank ℓ sublattices of $\mathcal{L}$ to rank $(n-\ell)$ sublattices of $\mathcal{L}^*$, preserving the approximation factor!

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Warm-up: $\ell = 1$ (SVP $\Rightarrow$ SVP)

- Plan:
  1. $w \leftarrow A(\mathcal{L}^*)$
  2. $\mathcal{L}' := \mathcal{L} \cap w^\perp$
  3. Output $A(\mathcal{L}')$.

- Base case: if $\mathrm{rank}(\mathcal{L}) = k$, output $\mathrm{SVP}(\mathcal{L})$.

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- Plan:
  - Choose initial "depth parameter"

    $\tau = \tau(n) \geq 0.$

  1. $w \leftarrow A(\mathcal{L}^*, \tau - 1)$
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  3. Output $A(\mathcal{L}', \tau)$

- Base cases:

  if $\tau = 0$, output $\mathrm{LLL}(\mathcal{L}, 1)$

  if $\mathrm{rank}(\mathcal{L}) = k$, output $\mathrm{SVP}(\mathcal{L})$.

Analysis

1. $w \leftarrow A(\mathcal{L}^*, \tau - 1)$
2. $\mathcal{L}' := \mathcal{L} \cap w^\perp$ (Dual)
3. Output $A(\mathcal{L}', \tau)$ (Composition)

Analysis

1. $\omega \leftarrow A(\mathcal{L}^*, \tau - 1)$
2. $\mathcal{L}' := \mathcal{L} \cap \omega^\perp$ (Dual)
3. Output $A(\mathcal{L}', \tau)$ (Composition)

$$\gamma(n, \tau) \leq \gamma(y, \mathcal{L}') \cdot \gamma(\mathcal{L}', \mathcal{L})^{\frac{1}{n-1}}$$

(Compositionality)
$l = 1, \quad m = n-1$

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$$= \gamma(y, \mathcal{L}') \cdot \gamma(w, \mathcal{L}^*)^{\frac{1}{n-1}}$$ (Duality)

$$= \gamma(n-1, \tau) \cdot \gamma(n, \tau - 1)^{\frac{1}{n-1}}.$$ (By definition)

Analysis

$$\boxed{\gamma(n,\tau) \le \gamma(n-1,\tau)\cdot\gamma(n,\tau-1)^{\frac{1}{n-1}}}$$

$$\boxed{\gamma(n,0) = 2^n} \qquad \boxed{\gamma(k,\tau) = \sqrt{k}}$$

- Can check that by induction

$$\boxed{\gamma(n,\tau) \le k^{\frac{n-1}{2(k-1)}} \cdot \exp\left(n^3/2^\tau\right)}$$

- Taking $\tau = O(\log n)$ recovers block reduction:

$$\boxed{\gamma = (1+o(1))\, k^{\frac{n-1}{2(k-1)}}}$$

Analysis

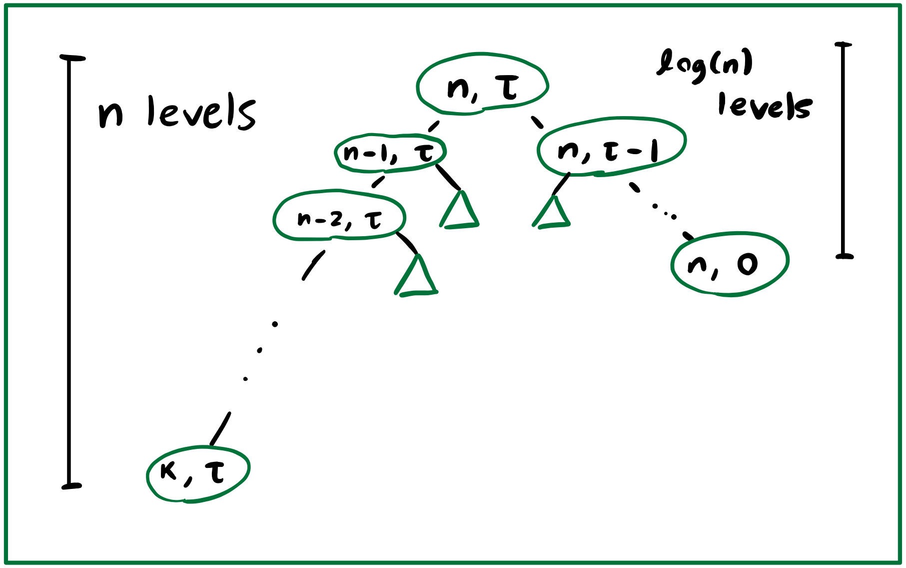
- SVP oracle calls dominate runtime.

$$\boxed{C(n, \tau) = C(n-1, \tau) + C(n, \tau-1)}$$

$$\boxed{C(k, \tau) = 1} \qquad \boxed{C(n, 0) = 0}$$

$$\boxed{C(n, \tau) = \binom{n-k+\tau-1}{\tau} \approx n^{\tau} = n^{O(\log n)}}$$

- Issue: Call tree highly unbalanced.



Take Two: $\ell > 1$ (DSP $\Rightarrow$ DSP)

- Plan:
  - Choose initial $\tau = O(\log n)$, $0 < \varepsilon < 1$.

1. $\hat{\mathcal{L}} \leftarrow A(\mathcal{L}^*, \varepsilon n, \tau - 1)$
2. $\mathcal{L}' := \mathcal{L} \cap (\hat{\mathcal{L}})^\perp$ // $\text{rank}(\mathcal{L}') = (1-\varepsilon)n$
3. Output $A(\mathcal{L}', \ell, \tau)$

- Base cases:

if $\tau = 0$, output $\text{LLL}(\mathcal{L}, \ell)$

if $\text{rank}(\mathcal{L}) = k$, output $\text{DSP}(\mathcal{L}, \ell)$.

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$\boxed{\text{What if } \mathrm{rank}(\mathcal{L}') = (1-\varepsilon)n < \ell\text{? Can't find a larger-rank sublattice!}}$

Take Two: $\ell > 1$ (DSP $\Rightarrow$ DSP)

- Plan:
  - Choose initial $\tau = O(\log n)$, $0 < \varepsilon \ll 1$.

  0. If $\ell > n/2$,
     output $\mathcal{L} \cap (A(\mathcal{L}^*, n-\ell, \tau))^\perp$.
  1. $\hat{\mathcal{L}} \leftarrow A(\mathcal{L}^*, \lceil \varepsilon n \rceil, \tau - 1)$
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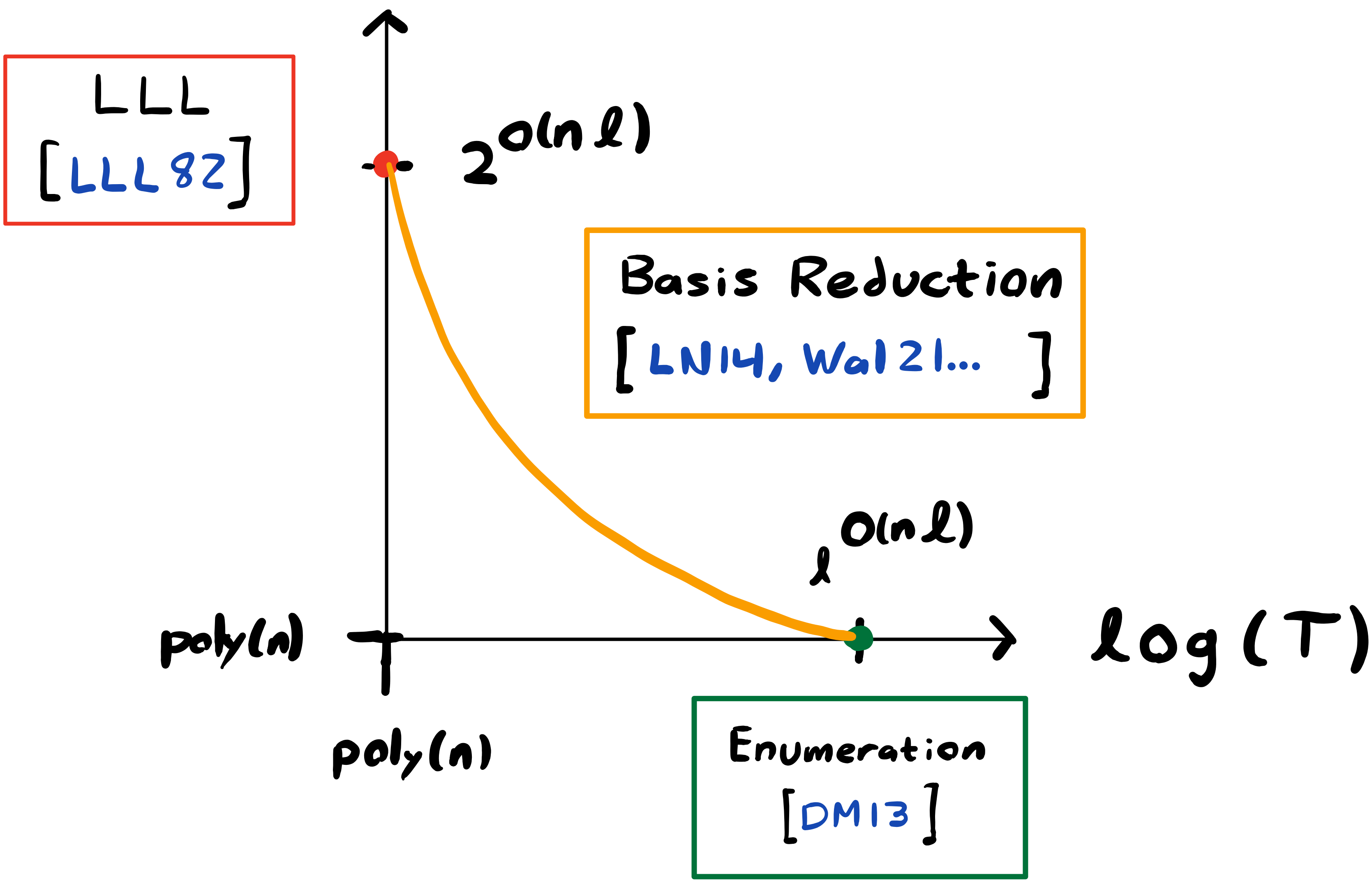
Time / γ Tradeoffs for γ-DSP_ℓ

γ-$\mathrm{DSP}_\ell(\mathcal{L})$: Find $\mathcal{L}' \subset \mathcal{L}$ of rank ℓ such that $\det(\mathcal{L}') \leq \gamma \cdot \det(\mathcal{L})^{\ell/n}$.



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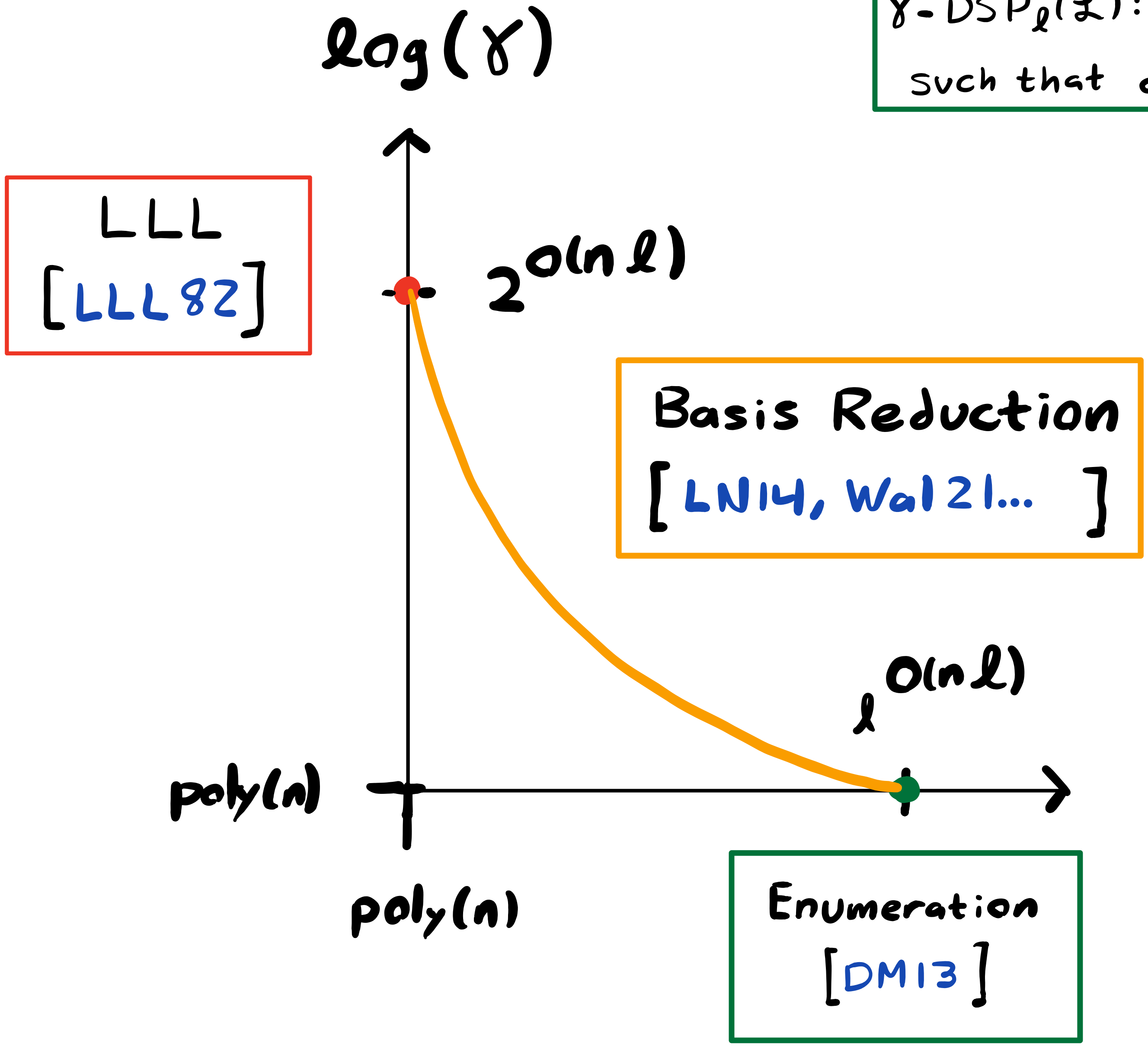

- Works by solving exact DSP in dimension k
- $T = \mathrm{poly}(n)\, \ell^{O(k\ell)}$
- $\gamma = \delta_{k,\ell}^{\frac{n-\ell}{2(k-\ell)}} = k^{\Theta\left(\frac{\ell(n-\ell)}{(k-\ell)}\right)}$.
- Requires $\ell \le k$.

Analysis

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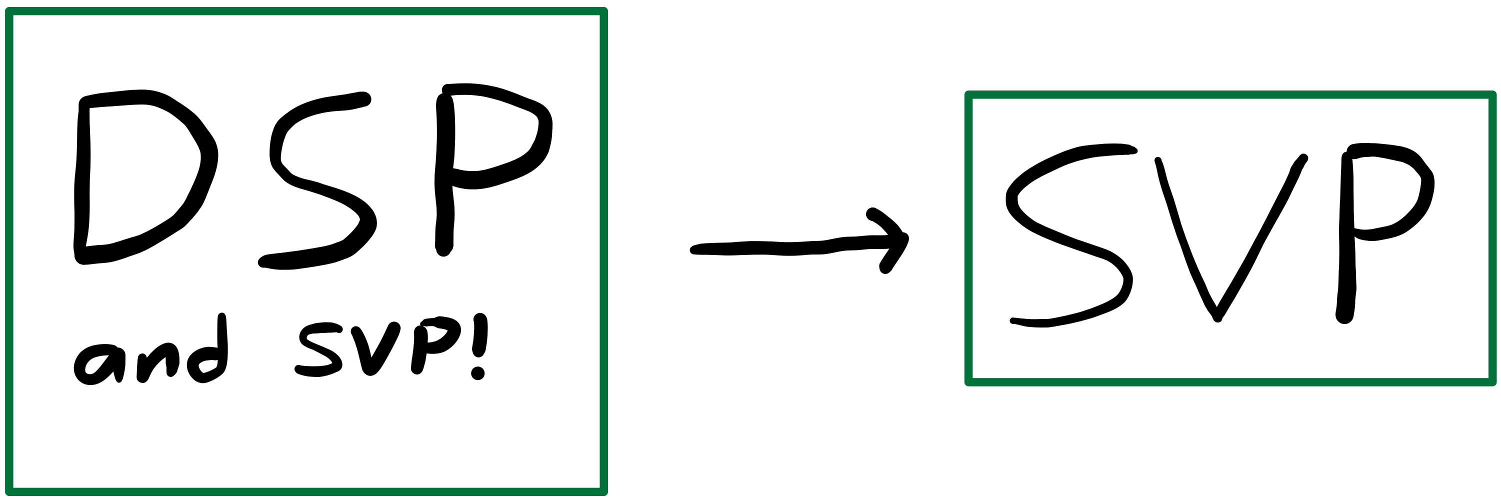
- Relies on conjecture $\sqrt{\delta_{k,\ell}} \approx k^{\frac{\ell(k-\ell)}{2(k-1)}}$
- DSP oracle calls are expensive.


DSP
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SVP

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- Solution idea: When ℓ is large, be patient:
  don't reduce the depth parameter τ.

The Reduction

Start with $\ell \leq n-k+1$.

- Duality: if $\max\{1, \frac{(n-k)}{5}\} < \ell < \frac{n}{2}$

  OR $\ell \geq n - \max\{1, \frac{(n-k)}{10}\}$:

  Output $\mathcal{L} \cap A(\mathcal{L}^*, n-\ell, \tau)^{\perp}$.

- Recursive step:

  $\hat{\mathcal{L}} \leftarrow A(\mathcal{L}^*, \lceil \frac{(n-k)}{20} \rceil, \tau - b)$

  Output $A(\mathcal{L} \cap (\hat{\mathcal{L}})^{\perp}, \ell, \tau)$.

  $$b = \begin{cases} 1, & \ell > \frac{n}{2} \\ 0, & \ell \leq \frac{n}{2} \end{cases}$$

- Base cases:

  if $\tau = 0$, output $LLL(\mathcal{L}, \ell)$.

  if $\text{rank}(\mathcal{L}) = k$ AND $\ell = 1$, return $SVP(\mathcal{L})$.

Analysis: Key Lemmas

All can be verified directly from local checks!

Lemma 1. All recursive calls satisfy

$$\min\{\ell, n-\ell\} \le n-k+1.$$

↳ Algorithm does not get "stuck".

Lemma 2. The potential $\Phi(n,\tau) := \tau + 20\log(n-k+1)$ drops by at least 1 from parent to grandchild.

↳ poly(n) runtime (oracle calls).

Lemma 3. The guess $f(n,\ell,\tau) := k^{\frac{\ell(n-\ell)}{2(k-1)}} \cdot \exp(n^3/2^\tau)$ satisfies the recurrence induced by $\mathcal{A}$.

↳ Achieve basis reduction tradeoff $k^{\frac{\ell(n-\ell)}{2(k-1)}}$.

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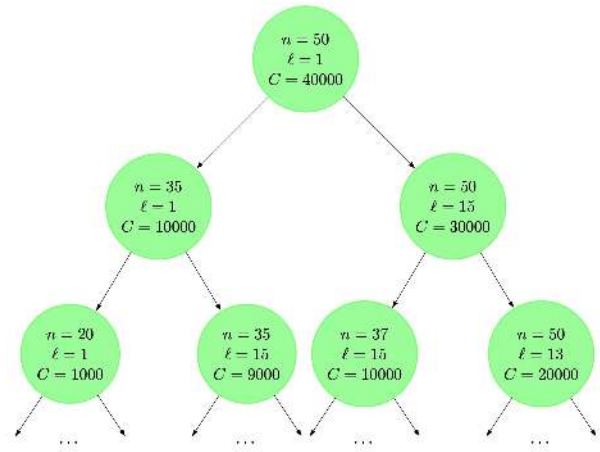
- Using dynamic programming, solve

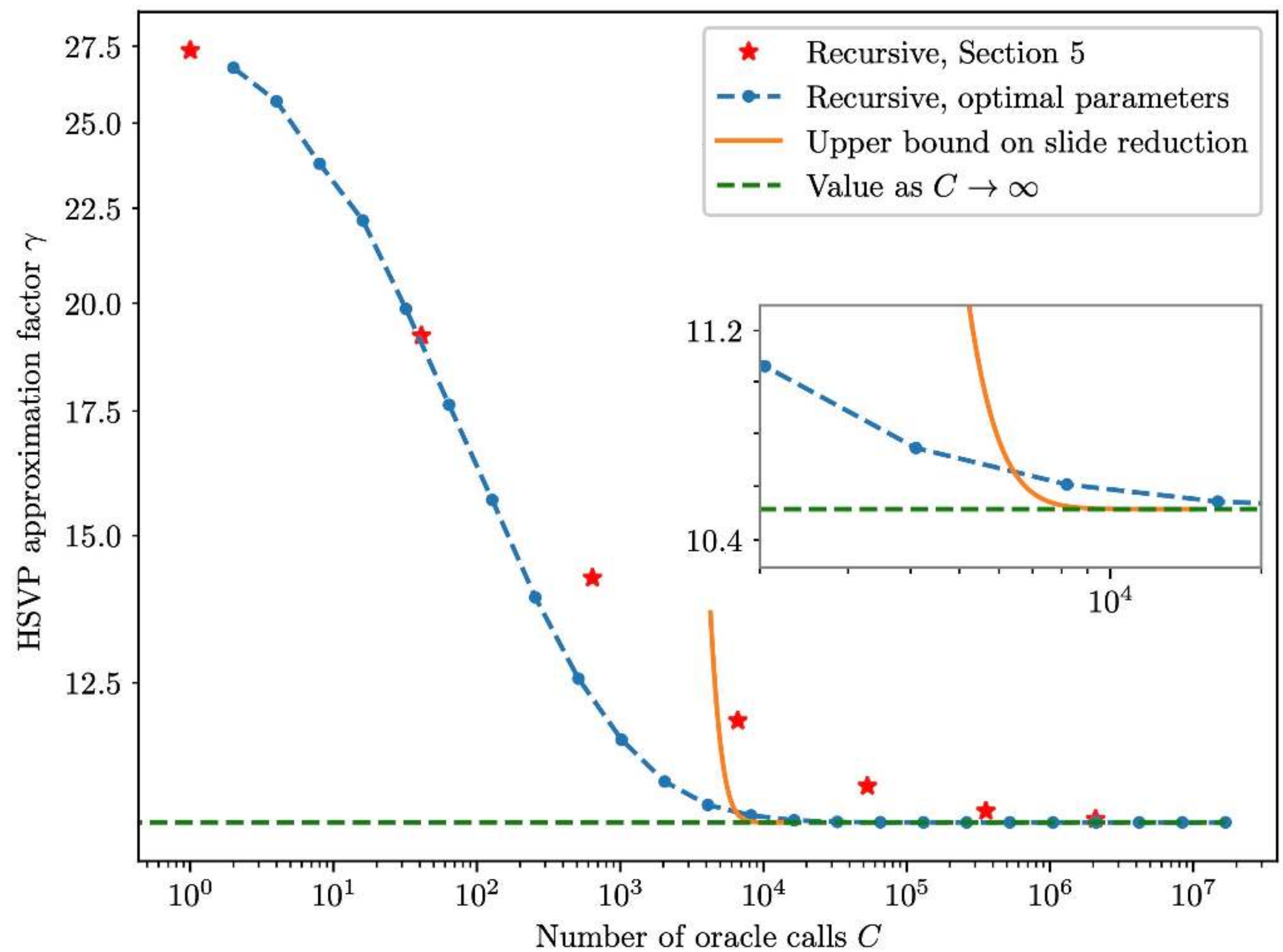
$$\gamma(n, \ell, c) := \min \left\{ \begin{array}{l} \gamma(n, n-\ell, c) \\ \displaystyle\min_{1 \le \ell^* \le n-k} \ \min_{c^* \le c} \ \gamma(n-\ell^*, \ell, c-c^*)\, \gamma(n, \ell^*, c^*)^{\frac{\ell}{n-\ell^*}} \end{array} \right\}$$

Results

- Optimal DP solution bounds rather massively improve on DSP → SVP guarantee.



n = 50, K = 10



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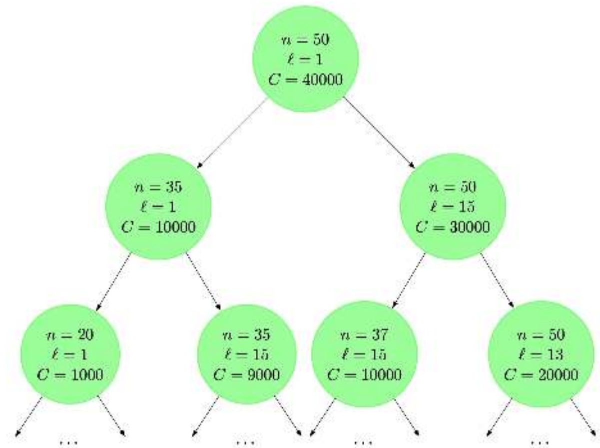

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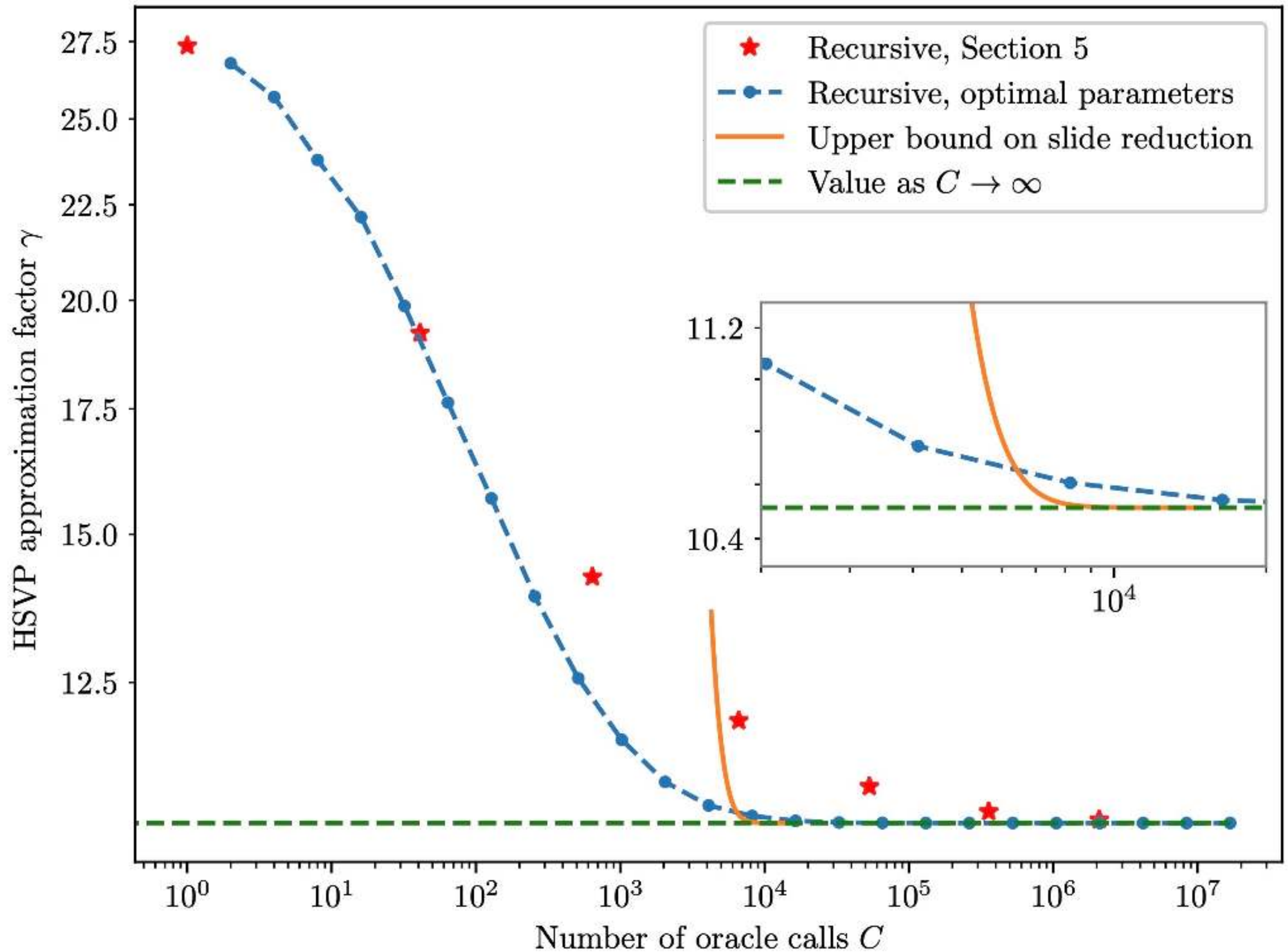

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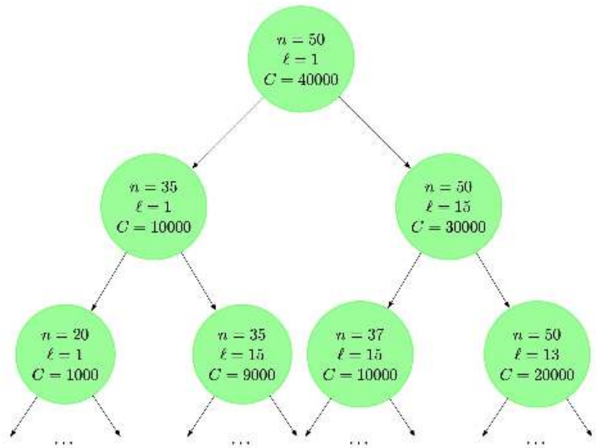

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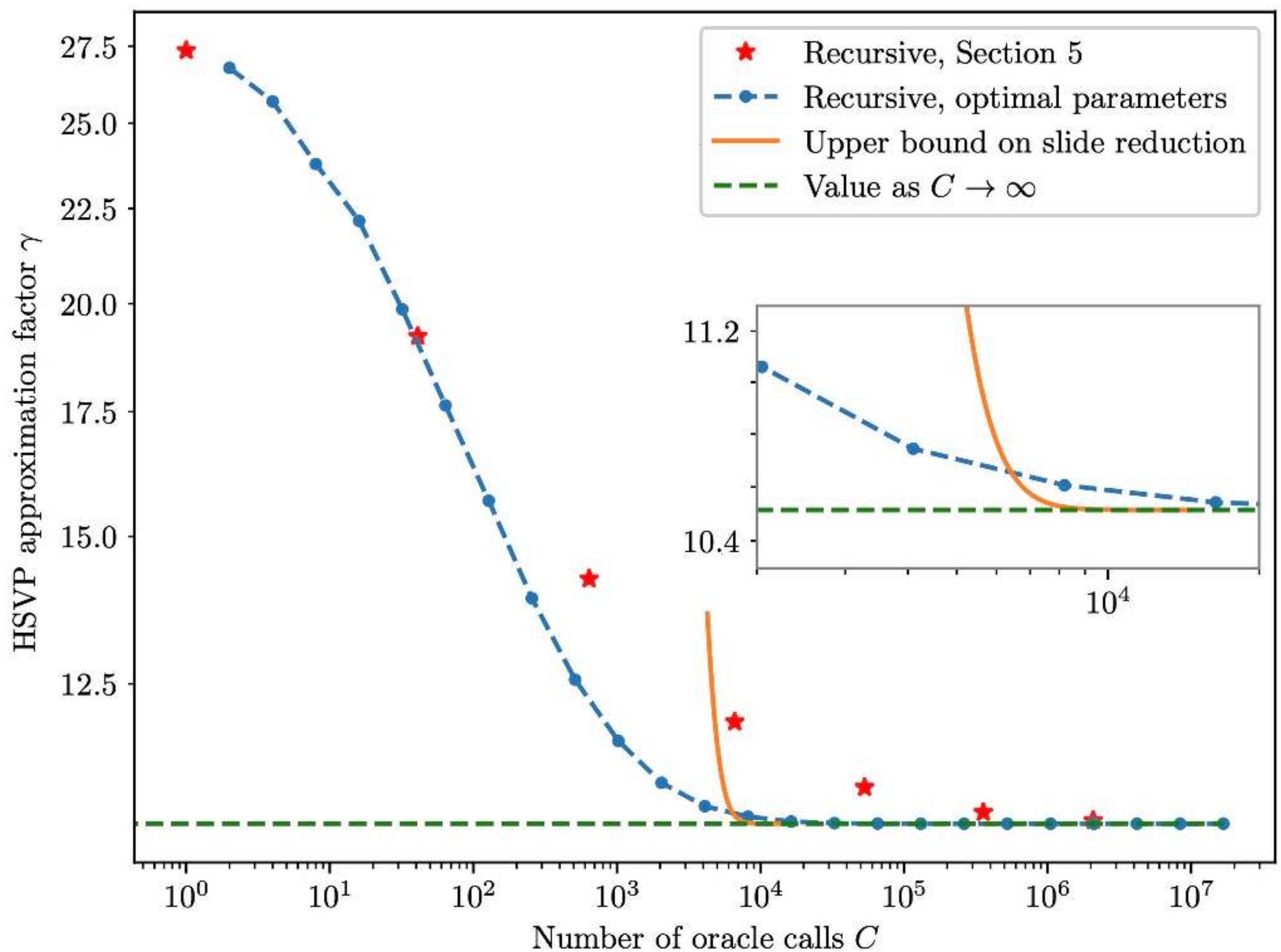

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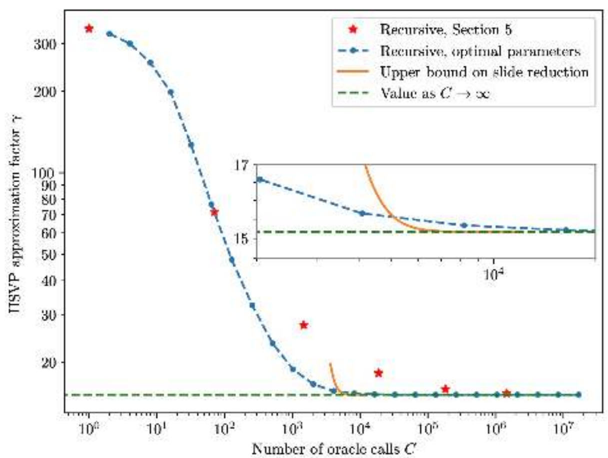

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- Rough estimate: solving SVP in dimension n takes time 2^n.

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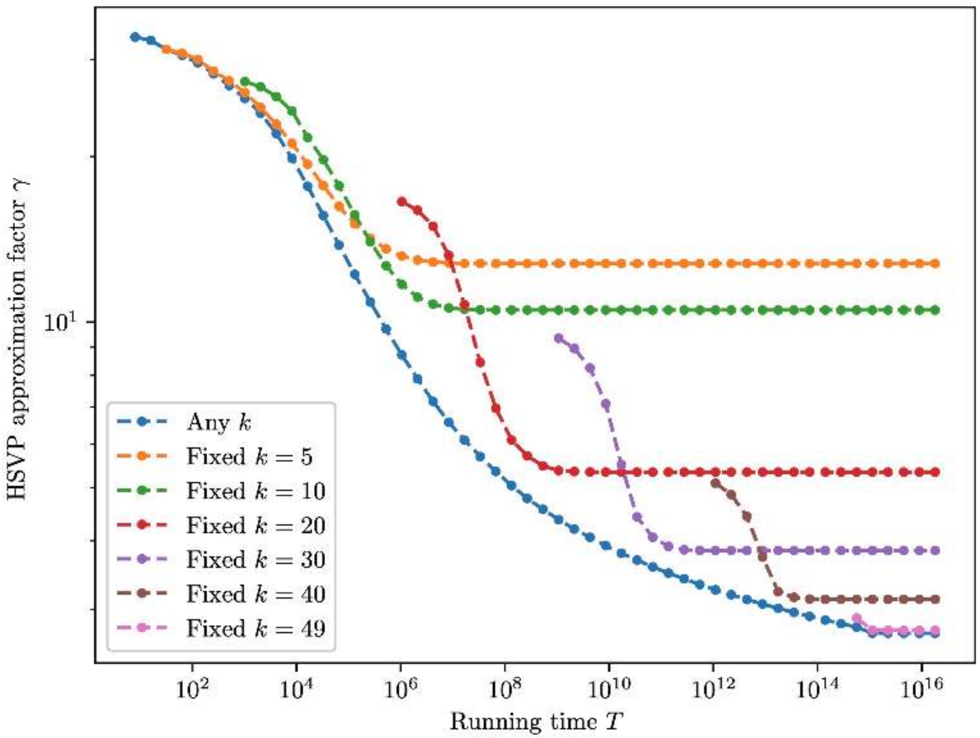

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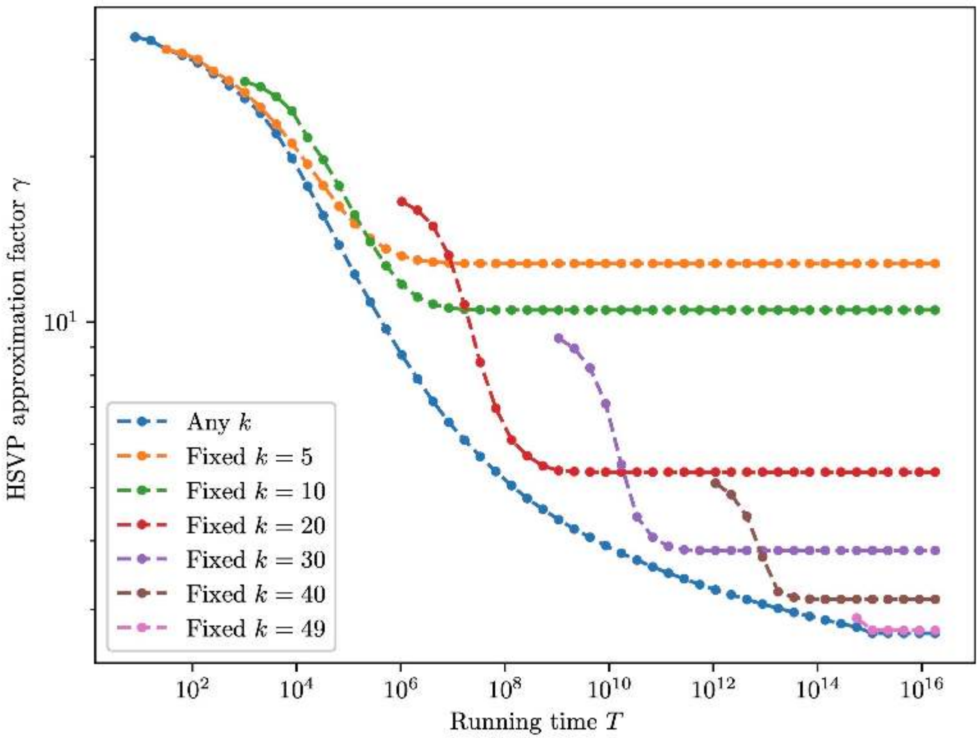

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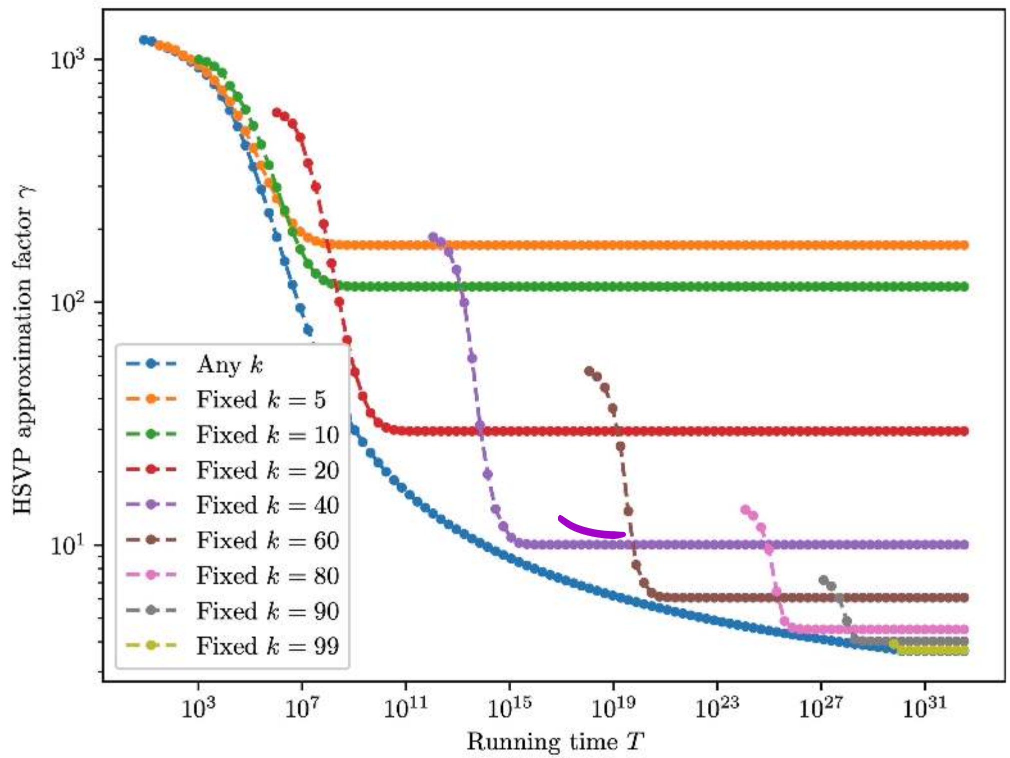

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- Study reductions that use very few oracle calls.

Thanks for listening!